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A FUNCTIONAL EQUATION FOR THE SINE

BY EDWARD B. VAN VLECK*

IN 1821 Cauchy† proved that the only real continuous solutions of the equation

$$\phi(x + y) + \phi(x - y) = 2\,\phi(x)\,\phi(y),$$

other than the trivial solutions $\phi(x) \equiv 0$ or 1, are the functions $\phi(x) = \cos ax$ and $\phi(x) = \cosh ax$. As analytic functions these two solutions are one and the same, but considered as real functions of a real variable they are distinct. The object of this note is to give a functional equation satisfied uniquely by the cosine, or, what amounts to the same thing, one satisfied uniquely by the sine. It appears to me much the simplest functional definition yet given,‡ and the fundamental properties of the solution follow from the equation with very great rapidity. The only assumption concerning its character is that it is real and continuous.

To define the sine I use the equation

$$(I) \qquad f(x - y + A) - f(x + y + A) = 2f(x)f(y),$$

in which A is a real constant. The following properties immediately result.

1) Put $x = y = 0$. Then $f(0) = 0$.

2) Replace y by $-y$. The left hand member of (I) changes sign, and therefore $f(y) = f(-y)$. In other words, $f(x)$ is an odd function.

3) Exchange x and y. It follows that

$$f(x - y + A) = f(y - x + A) = -f(x - y - A).$$

Hence the addition of $2A$ to the argument changes the sign of the function, and therefore $4A$ is a period of $f(x)$. Without loss of generality we may henceforth suppose A to be positive.

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* Presented to the American Mathematical Society (Chicago).

† *Analyse Algebrique*, p. 114; *Oeuvres* ser. 2, vol. 3, p. 106.

‡ For other functional definitions see Moore, ANNALS OF MATHEMATICS, (1), vol. 9 (1895), p. 43; Lunn, Ibid (2), vol. 10 (1908), p. 37; and Osgood, *Lehrbuch der Funktionentheorie*, vol. 1, p. 510.

4) Replace x by $x + A$ and y by $y + A$. Equation (I) then gives with the aid of 3),

$$(1) \qquad f(x - y + A) + f(x + y + A) = 2f(x + A)\, f(y + A).$$

Put now $\phi(u) \equiv f(u + A)$. Then

$$\phi(x - y) + \phi(x + y) = 2\phi(x)\, \phi(y).$$

Thus $f(x + A)$ satisfies Cauchy's functional equation, and by 3) it must be the periodic solution of that equation. Selecting this solution, we have from Cauchy's work

$$f(x) = \cos \frac{\pi}{2A}(x - A) = \sin \frac{\pi x}{2A}.$$

The discussion can, however, be carried forward without having recourse to Cauchy's equation. Thus

5) If we put $x = y = A$ in (I), we have

$$f(A) + f(A) = 2f^2(A),$$

whence $f(A) = 0$ or 1. But if $f(A) = 0$, it follows from (I) by putting $y = A$ that $f(x)$ identically vanishes. Putting aside this trivial solution, we must take $f(A) = 1$.

6) Put $y = x + A$. Then from (I),

$$f(2x) = 2f(x)\, f(x + A).$$

This equation was used by Moore in his determination of the sine, but supplementary conditions of equal importance and restrictive scope were necessarily added to obtain a definite function.

7) By placing $x = y$ in both (I) and (1) and adding we obtain

$$(2) \qquad 1 = f^2(x) + f^2(x + A).$$

8) It may be shown next that $f(x)$ is positive and steadily increases when x increases from 0 to A.

For if possible, let us suppose that x vanishes for some value B between these limits. If then we set $y = B$ in (I), we obtain

$$f(x - B + A) = f(x + B + A).$$

Consequently $2B$ must be a period of $f(x)$. There must be a smallest B, since otherwise the function would have a period as small as we please and, being continuous, would therefore be constant. I shall use this smallest B.

For $x = y$, equation (I) becomes

$$1 - f(2x + A) = 2f^2(x).$$

Hence $|f(2x + A)| < 1$ for values of x between 0 and B. It follows then by replacing x by $2x$ in (2) that $|f^2(2x)| > 0$ for the same range of values of x. But this gives a contradiction, inasmuch as $f(2x)$ by our hypothesis must vanish for $x = B/2$. Consequently $f(x)$ can not vanish for $0 < x \leqq A$. Since also $f(A) = 1$, it must be positive.

Furthermore, if we put $x = 0$ in (I) and then replace y by x, we have

$$f(A + x) = f(A - x).$$

This shows that $f(x)$ is also positive for $A < x < 2A$.

To see finally that $f(x)$ steadily increases with x in the interval $(0, A)$, replace x by $x + h + A$ and y by h in (I). The equation becomes

$$f(x + 2h) - f(x) = 2f(x + h + A)f(h). \tag{3}$$

For positive $h < A$ and for $0 < x + h < A$ the right hand member is positive, and therefore any small positive increment $2h$ in the argument results in an increase of $f(x)$ as long as x lies in the interval $(0, A)$.

9) From (3) we get the difference quotient

$$Q(x + 2h, x) \equiv \frac{f(x + 2h) - f(x)}{2h} = f(x + h + A)\frac{f(h)}{h}. \tag{4}$$

This shows that *if $f(h)/h$ has a limit c for $h = 0$, $f(x)$ will have a derivative, of which the value is $cf(x + A)$.*

10) A proof of the existence of a limit for $f(h)/h$ when h approaches 0 has been given by Osgood,* and his proof can be based upon the equations

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* Loc. cit., pp. 512–513.

already here obtained. Another simple proof, based directly upon a consideration of the difference quotient, is as follows.

Putting $x = 0$ in (4) we see that

$$\frac{f(2h)}{2h} < \frac{f(h)}{h}.$$

If then x approaches 0 over any set of positive values $2h$, h, $h/2$, $h/4, \cdots$, the ratio $f(x)/x$ will steadily increase.

We shall next prove that it can not increase beyond all limit. Divide the interval $(0, 2h)$ into 2^n equal parts of length $h/2^{n-1}$. The difference quotient taken for any two consecutive points of division, $p - h/2^{n-1}$ and p, has by (4) the value

$$Q\left(p, p - \frac{h}{2^{n-1}}\right) = f\left(p - \frac{h}{2^n} + A\right) \cdot \frac{f\left(\frac{h}{2^n}\right)}{\frac{h}{2^n}}.$$

Hence since $f\ (x + A)$ decreases in the x-interval $(0, A)$ we have the inequality

$$f(A + 2h)\cdot \frac{f\left(\frac{h}{2^n}\right)}{\frac{h}{2^n}} < Q\left(p, p - \frac{h}{2^{n-1}}\right) < \frac{f\left(\frac{h}{2^n}\right)}{\frac{h}{2^n}}.$$

Now if (a, b) is any interval of the x-axis and c is an internal point of division, by a very fundamental but simple theorem* regarding the difference quotient, $Q(a, b)$ is intermediate in value between $Q(a, c)$ and $Q(c, b)$. By obvious extension, if the interval is divided into any number of parts by the points $c_0 = a, c_1, c_2, \cdots, c_{n-1}, c_n = b$, the value of $Q(a, b)$ will lie between the greatest and least values of $Q(c_i, c_{i+1})$. Consequently

$$(5) \qquad f(A + 2h)\ \frac{f\left(\frac{h}{2^n}\right)}{\frac{h}{2^n}} < Q(p, 0) < \frac{f\left(\frac{h}{2^n}\right)}{\frac{h}{2^n}}.$$

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* Cf. Baire, *Annali di Matematica*, (3) vol. 3 (1899), p. 103.

In this inequality let n increase indefinitely. If $f(x)/x$ increases without limit when x traverses the values $h/2$, $h/4$, $h/8$, . . . it will follow at once that $Q(p, 0)$ is infinite. As this is impossible, we conclude that $f(x)/x$ for this set of values increases to a finite limit c.

Consider now the dense set, S, consisting of all points of division of the interval $(0, 2h)$ which are obtained by taking successively $n = 1, 2, 3, \cdots$. Let p be any point of the set. By increasing n indefinitely in (5) we obtain

$$cf(2h + A) \leqq Q(p, 0) \equiv \frac{f(p)}{p} < c. \tag{6}$$

We can contract the interval $2h$ as much as we please by replacing h by $h/2^n$. It follows then directly from (6) that $f(x)/x$ has the limit c when x approaches 0 over the set S which is everywhere dense in the interval $(0, 2h)$.

Now if this is true for a *continuous* $f(x)$, the conclusion can be extended immediately from the dense set S to the interval itself. For if x' is any point of the interval not included in S, a point p of S can be found so near to x' that

$$\frac{f(x')}{x} = \frac{f(p)}{p}(1 + \epsilon),$$

where ϵ is as small as we wish. Consequently $f(x')/x'$ has a common limit with $f(p)/p$. Since $f(x)$ is odd, the limit is the same for negative h.

The essential properties of $f(x)$, including its differentiability, have been established, from which its identification with $\sin cx$ readily follows. This can be done in various ways. Thus, for example, if the derivative

$$\frac{df(x)}{dx} = cf(x + A) \equiv c\sqrt{1 - f^2(x)}$$

is used for this purpose, it follows immediately by integration that $f(x) = \sin cx$.

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